

# THE EXACTNESS OF A GENERAL SKODA COMPLEX

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## Abstract

We show that a  $q$ -th Skoda complex with a general plurisubharmonic weight function is exact if  $q$  is sufficiently large. Following work of Lazarsfeld and Lee, this implies that not every integrally closed ideal can be written as a multiplier ideal even if we allow nonalgebraic plurisubharmonic weights. This general case of the exactness uses a very different method from the algebraic case, due to the openness conjecture of Demailly and Kollár.

## 1 Introduction

Let  $X$  be a smooth algebraic variety over complex numbers. In complex algebraic geometry, a *pole* or a *singular weight* of the form  $\frac{1}{|f|^2}$  plays an important role, where  $f$  is a holomorphic section of a line bundle. It is natural to generalize a little bit and consider a singular weight  $e^{-\varphi}$  where  $\varphi$  is a plurisubharmonic function, say, on a connected open subset  $\Omega \subset X$ . (See [D, Sec.1.B.] for the definition and basic properties of a plurisubharmonic function, or a *psh* function for abbreviation from now on.) Starting from  $e^{-\varphi}$ , there are two fundamental ways to define an ideal sheaf of local holomorphic function germs  $u$ : collecting those with  $|u|^2 e^{-\varphi}$  locally bounded above, on one hand and collecting those with the integral  $\int_{\Omega} |u|^2 e^{-\varphi}$  finite, on the other hand. The former gives an *integrally closed* ideal and the latter is called a *multiplier* ideal.

A multiplier ideal is always an integrally closed ideal, but it had been a question whether the two classes of ideals were really different until [LL] proved so when the psh  $\varphi$  is *algebraic*, e.g. of the form  $\log |f|^2$  for  $f$  holomorphic. The proof of [LL] uses the exactness of a *Skoda complex* - a Koszul-type complex of sheaves involving multiplier ideals in a natural way (see Definition 2.2). For the special algebraic case of  $\varphi = \log |f|^2$ , the exactness is a rather elementary consequence of a local vanishing theorem [L, (9.4.4), (9.6.36)].

However, the general case turns out to be surprisingly nontrivial because of the openness conjecture of [DK] for a general ‘transcendental’ psh  $\varphi$ . (If a psh function is not *algebraic*, or more precisely, does not have *analytic singularities* in the sense of [D, Definition 1.10], we will say it is a *transcendental* psh function. ) The openness conjecture is known only for the case of  $\dim X = 2$  by [FJ]. We deal with the difficulty given by the openness conjecture for general dimension by using a generalized Skoda type division in the place of the vanishing theorem.

The relevance of a psh function to algebraic geometry is well illustrated by the fact that a line bundle  $L$  on a smooth projective variety  $X$  is pseudoeffective if and only if  $L$  carries a singular hermitian metric whose local weight functions are psh. If  $L$  is moreover effective, we can take the local weight functions as algebraic ones, or more precisely those having analytic singularities. When  $L$  is the canonical line bundle of  $X$ , the abundance conjecture predicts that  $K_X$  being pseudoeffective implies  $K_X$  being effective. That somehow indicates the distance between a transcendental psh function and a psh function with analytic singularities.

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Another indication of the subtlety of a transcendental psh function is given by the openness conjecture of [DK] which expects that a transcendental psh function has a similar behaviour as a psh function with analytic singularities in terms of their multiplier ideal sheaves. In fact, we show that the following modified version of the openness conjecture implies the exactness of a Skoda complex we want. Let  $\Omega \subset \mathbf{C}^n$  be a Stein open subset and  $\alpha, \beta$  psh functions on  $\Omega$ . There is the stationary limit ideal sheaf  $\mathcal{J}_+(\alpha, \beta)$  among the multiplier ideal sheaves  $\mathcal{J}(\alpha + t\beta)$  as  $t \rightarrow 0$ . The modified version of the openness conjecture is that  $\mathcal{J}_+(\alpha, \beta) = \mathcal{J}(\alpha)$  which implies the openness conjecture of [DK].

Apart from the use for local syzygy in [LL], a Skoda complex was originally used to prove the Skoda-type division theorem in an algebraic way (using cohomology vanishing) in [EL]. On the other hand, the original analytic way of [Sk72] to prove the Skoda-type division theorem (not via cohomology vanishing) does not involve the use of a Skoda complex. Interestingly, however, [H67] had used a Koszul complex (together with his  $L^2$  methods for  $\bar{\partial}$ ) for a prototype result toward Skoda division. Later it was superseded by the more refined  $L^2$  methods of [Sk72].

Inspired by [H67], we apply the  $L^2$  methods of [Sk72] to a Skoda complex setting and prove

**Theorem 1.1.** *Let  $X$  be a complex manifold and  $L$  and  $M$  line bundles on  $X$ . Let  $e^{-\psi}$  be a singular hermitian metric with psh weight for the line bundle  $L$ . Let  $g_1, \dots, g_p \in H^0(X, L)$ . Then there exists an integer  $q \geq p$  such that the  $q$ -th Skoda complex associated to  $g_1, \dots, g_p$  is exact. More precisely, one can take  $q = \lfloor \frac{1}{4}p^2 + \frac{1}{2}p + \frac{5}{4} \rfloor$ .*

Theorem 1.1 can be considered as a generalized version of the Skoda type division theorem [Sk72] in that we *divide* not only at the right end of the Skoda complex but also at all the other intermediate terms of the Skoda complex. The value of  $q$  in the statement comes from  $\lfloor \frac{1}{4}p^2 + \frac{1}{2}p + \frac{5}{4} \rfloor = \max_{0 \leq m \leq p} m(p - m + 1) + 1$ . We believe the optimal value of  $q$  which can be used in the statement will be  $q = p$  as is also indicated by (3.4). But the present value of  $q$  is sufficient to establish the following extension of the main result of [LL].

**Corollary 1.2.** *There exist a complex algebraic variety  $X$  and an integrally closed ideal sheaf  $\mathfrak{b}$  on it such that  $\mathfrak{b}$  cannot be written as a multiplier ideal sheaf  $\mathcal{J}(\varphi)$  even if we allow  $\varphi$  to be a general plurisubharmonic function.*

This paper is organized as follows. In Section 1, we motivate and give the definition of a Skoda complex. In Section 2, we give a version of the openness conjecture of [DK] and show that it implies the exactness of a Skoda complex in full generality. In Section 3, we review Hörmander's  $L^2$  estimates for  $\bar{\partial}$  and its initial use by Hörmander and Skoda for the division problem. In Section 4, we give the proof of our main theorem (1.1). In Section 5, we follow [LL] to derive the existence of an integrally closed ideal that is not a multiplier ideal (1.2).

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## 2 Definition of a Skoda complex

Let  $X$  be a complex manifold and  $L$  a line bundle on  $X$ . Let  $g_1, \dots, g_p \in H^0(X, L)$  be holomorphic sections. Let  $M$  be another line bundle. Let  $\mathcal{A}(M)$  denote either the set of all holomorphic sections of  $M$  or the set of all complex-valued measurable sections of  $M$ . Given  $u \in \mathcal{A}(M)$ , we can ask whether there exist  $h_1, \dots, h_p \in \mathcal{A}(M - L)$  such that  $u = h_1 g_1 + \dots + h_p g_p$ . Such a division problem is concerned with the surjectivity of the multiplication map  $P : \mathcal{A}(M - L)^{\oplus p} \rightarrow \mathcal{A}(M)$  given by  $(v_1, \dots, v_p) \mapsto v_1 g_1 + \dots + v_p g_p$  from the direct sum of  $p$  copies of  $\mathcal{A}(M - L)$  on the left. In some approaches to the division problem, one needs to extend the single map  $P$  to a Koszul-type complex to the left:

$$0 \rightarrow \mathcal{A}(M - pL)^{\oplus \binom{p}{p}} \rightarrow \dots \rightarrow \mathcal{A}(M - 2L)^{\oplus \binom{p}{2}} \rightarrow \mathcal{A}(M - L)^{\oplus p} \rightarrow \mathcal{A}(M) \rightarrow 0.$$

where we use the basis  $\{e_{i_1} \wedge \dots \wedge e_{i_m} \mid 1 \leq i_1 < \dots < i_m \leq p\}$  for  $\mathcal{A}(M - mL)^{\oplus \binom{p}{m}} =: \mathcal{B}_m$  and the map  $P : \mathcal{B}_m \rightarrow \mathcal{B}_{m-1}$  in the complex is the usual Koszul map

$$P(e_{i_1} \wedge \dots \wedge e_{i_m}) = \sum_{k=1}^m (-1)^{k-1} g_{i_k} e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_m}. \quad (1)$$

$P$  is defined by (1) and the linearity in  $\mathcal{A}$ . In effect, we introduced  $e_i$ 's in order to define the map  $P$ . Now it is also natural to consider the weight  $\varphi = \log |g|^2$  (defining  $|g|^2 := |g_1|^2 + \dots + |g_p|^2$  throughout this paper) and (for  $1 \leq m \leq p$ ) the subset  $\mathcal{A}(M - mL, e^{-(p-m)\varphi} e^{-\psi}) \subset \mathcal{A}(M - mL)$  of sections that are square-integrable with respect to  $e^{-(p-m)\varphi} e^{-\psi} dV$  where  $dV$  is a volume form and  $e^{-\psi}$  is a (auxiliary) weight for the line bundle  $(M - pL)$ .

**Proposition 2.1.** *The restriction of the Koszul map  $P$  to  $\mathcal{A}(M - mL, e^{-(p-m)\varphi} e^{-\psi})^{\oplus \binom{p}{m}}$  has its image contained in  $\mathcal{A}(M - (m-1)L, e^{-(p-m+1)\varphi} e^{-\psi})^{\oplus \binom{p}{m-1}}$ .*

*Proof.* Let  $u \in \mathcal{A}(M - mL, e^{-(p-m)\varphi} e^{-\psi})^{\oplus \binom{p}{m}}$  and write it as  $u = \sum_J u_J e_J$  where the index  $J$  denotes  $(j_1, \dots, j_m)$  with  $1 \leq j_1 < \dots < j_m \leq p$ . We know that for each index  $J$ ,  $\int_X |u_J|^2 e^{-(p-m)\varphi} e^{-\psi} dV < \infty$ . Consider  $P(u)$  and its  $I$ -th component where the index  $I$  denotes  $(i_1, \dots, i_{m-1})$  with  $1 \leq i_1 < \dots < i_{m-1} \leq p$ . Call the  $I$ -th component as  $\sigma$ . Then

$$\sigma = \sum_{t \notin I, 1 \leq t \leq p} g_t u_{I \cup t}$$

where the index  $I \cup t$  of  $u_{I \cup t}$  denotes the rearrangement in the right order,  $|I \cup t|$  being  $m$ . Now the conclusion follows from Cauchy-Schwarz:

$$\left| \sum_{t \notin I, 1 \leq t \leq p} g_t u_{I \cup t} \right|^2 e^{-(p-(m-1))\varphi} e^{-\psi} \leq |g|^2 \sum_J |u_J|^2 e^{-(p-(m-1))\varphi} e^{-\psi} = \sum_J |u_J|^2 e^{-(p-m)\varphi} e^{-\psi}.$$

□

Consequently we have the following Koszul-type complex:

$$\begin{aligned}
0 \rightarrow \mathcal{A}(M - pL, e^{-\psi})^{\oplus \binom{p}{p}} &\rightarrow \mathcal{A}(M - (p-1)L, e^{-(\psi+\varphi)})^{\oplus \binom{p}{p-1}} \rightarrow \dots \\
&\rightarrow \mathcal{A}(M - 2L)^{\oplus \binom{p}{2}} \rightarrow \mathcal{A}(M - L)^{\oplus p} \rightarrow \mathcal{A}(M) \rightarrow 0
\end{aligned}$$

Now if we restrict our attention to holomorphic sections and the corresponding sheaves, we get a complex of coherent sheaves:

**Definition 2.2** ([EL], [L, (9.6.36)]). Let  $X$  be a complex manifold and  $L, M$  line bundles on  $X$ . Let  $(M, e^{-\psi})$  be a singular hermitian metric. Let  $W$  be the vector space spanned by holomorphic sections  $g_1, \dots, g_p \in H^0(X, L)$ . Let  $q \geq p$ . The  $q$ -th **Skoda complex** ( $\text{Skod}_q$ ) is the complex of Koszul maps (which we will construct in the below)

$$\begin{aligned}
0 \rightarrow \Lambda^p V \otimes \mathcal{J}((q-p)\varphi + \psi) \otimes \mathcal{O}((q-p)L + M) &\rightarrow \dots \\
\dots \rightarrow \Lambda^m V \otimes \mathcal{J}((q-m)\varphi + \psi) \otimes \mathcal{O}((q-m)L + M) &\rightarrow \dots \\
\dots \rightarrow \Lambda^1 V \otimes \mathcal{J}((q-1)\varphi + \psi) \otimes \mathcal{O}((q-1)L + M) &\rightarrow \mathcal{J}(q\varphi + \psi) \otimes \mathcal{O}(qL + M) \rightarrow 0 \quad (2)
\end{aligned}$$

*Construction of (2).* Let  $f: Y \rightarrow X$  be a log-resolution of the ideal  $\mathfrak{a} \subset \mathcal{O}_X$  generated by  $g_1, \dots, g_p$  by Hironaka's theorem. Later in (3.4), we will use the fact that  $f$  is given by composition of blow-ups along smooth subvarieties. Let  $F$  be the exceptional divisor on  $Y$  such that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ . Consider the Koszul complex defined by pullbacks of generators of  $\mathfrak{a}$  where  $V$  is the vector space spanned by the pullbacks.

$$0 \rightarrow \Lambda^p V \otimes \mathcal{O}_Y(pF) \rightarrow \dots \rightarrow \Lambda^2 V \otimes \mathcal{O}_Y(2F) \rightarrow V \otimes \mathcal{O}_Y(F) \rightarrow \mathcal{O}_Y \rightarrow 0$$

Then twist through by a coherent sheaf  $\mathcal{O}_Y(K_{Y/X} - qF) \otimes \mathcal{J}(f^*\psi)$  and it stays exact since [EP, p.7, footnote2] says that: the Koszul complex is locally split and its syzygies are locally free, so twisting by any coherent sheaf preserves exactness. We get our Skoda complex by pushforwarding this exact sequence under  $f$  since (for  $0 \leq m \leq p$ )

$$f_*(\mathcal{O}_Y(K_{Y/X} - (q-m)F) \otimes \mathcal{J}(Y, f^*\psi)) = \mathcal{J}(\Omega, (q-m)\varphi + \psi).$$

This is from the change of variables formula [L, (9.3.43)] and the fact that  $\mathcal{O}_Y(-(q-m)F) \otimes \mathcal{J}(f^*\psi) = \mathcal{J}(f^*((q-m)\varphi + \psi))$  which in turn comes from comparing the two sides using holomorphic function germs satisfying the local integrability conditions of the multiplier ideal sheaves.  $\square$

When  $\varphi$  has *analytic* (or algebraic) *singularities* [D, Definition 1.10], the complex  $(\text{Skod}_q)$  is exact for all  $q \geq p$  by Theorem 9.6.36 [L]. In the general case, we will prove that it is exact for sufficiently large  $q \geq p$ .

### 3 Openness conjecture for plurisubharmonic functions

Suppose that a plurisubharmonic function  $e^{-\varphi}$  is given on a complex manifold  $X$ . Let  $0 \leq c < d$  be real numbers. If  $(e^{-\varphi})^d$  is  $L^1$ , then  $(e^{-\varphi})^c$  is  $L^1$  as well since  $(e^{-\varphi})^c \leq (e^{-\varphi})^d$ . So (fixing any compact set  $K \subset X$ ) the set  $T := \{c \geq 0 \mid e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } K\}$  is an interval. We call  $\sup T$  the **singularity exponent** of  $\varphi$  and write  $c_K(\varphi) := \sup T$ . Is the interval  $T$  open

at the right end? The openness conjecture of [DK] says so, that is,  $c_K(\varphi) \notin T$ . On the other hand, the following statement in terms of multiplier ideal sheaves is also natural to consider:

**Conjecture 3.1.** *Let  $\alpha$  and  $\beta$  be plurisubharmonic functions on a complex manifold. Let  $\mathcal{J}_+(\alpha, \beta)$  be the maximal element of  $\{\mathcal{J}(\alpha + t\beta) \mid t > 0\}$  as  $t \rightarrow 0$ . Then  $\mathcal{J}_+(\alpha, \beta) = \mathcal{J}(\alpha)$ .*

The special case  $\beta = \alpha$  gives the conjecture  $\mathcal{J}_+(\alpha) = \mathcal{J}(\alpha)$  (where  $\mathcal{J}_+(\alpha) := \mathcal{J}_+(\alpha, \alpha)$ ) which was considered in [D, (15.2.2)] and [DEL]. This special case of (3.1) implies the openness conjecture of [DK] though the converse is not known.

**Proposition 3.2.** *(3.1) implies the openness conjecture of [DK].*

*Proof.* Let  $c = c_K(\varphi)$  the singularity exponent. Take  $\alpha = c\varphi$  and  $\beta = \varphi$ . Suppose that  $c$  belongs to the interval  $T$ . Then  $e^{-2c\varphi}$  is  $L^1$ , so  $\mathcal{J}(c\varphi)$  is trivial while for any  $t > 0$ , we have  $\mathcal{J}(c\varphi + t\varphi)$  nontrivial. This contradicts (3.1).  $\square$

**Proposition 3.3.** *(3.1) is true if  $\alpha$  has analytic singularities (see [D, Definition 1.10]).*

*Proof.* The special case  $\alpha = 0$  is a result of Skoda ([Sk72], see [D, Lemma (5.6)]), which we will use. We need to show that  $\mathcal{J}(\alpha) \subset \mathcal{J}(\alpha + \delta\beta)$  for some  $\delta > 0$ . Let  $f$  be a holomorphic function germ which belongs to  $\mathcal{J}(\alpha)$ . Then  $\exp(\log |f|^2 - \alpha)$  is locally integrable. Since  $\alpha$  has analytic singularities, we can use a log-resolution of  $\alpha$  and  $\text{div}(f)$  to have  $\epsilon > 0$  such that  $\exp((1 + \epsilon)(\log |f|^2 - \alpha))$  is still integrable. Now choose  $p$  such that  $\frac{1}{1+\epsilon} + \frac{1}{p} = 1$ . Then we can choose  $\delta > 0$  such that  $(e^{-\beta})^{\delta p}$  is integrable from the special case  $\alpha = 0$  of Skoda: [D, (5.6) a] says that the multiplier ideal sheaf of  $\delta p\beta$  is trivial when the Lelong number of  $\delta p\beta$  is less than 1. Then it gives the finiteness of the first factor on the right in the following Hölder inequality:

$$\int_{\Omega} |f|^2 e^{-(\alpha + \delta\beta)} dV \leq \left( \int_{\Omega} e^{-\delta p\beta} dV \right)^{\frac{1}{p}} \left( \int_{\Omega} |f|^{2(1+\epsilon)} e^{-(1+\epsilon)\alpha} dV \right)^{\frac{1}{1+\epsilon}} < \infty.$$

$\square$

On the other hand, the special case of  $\beta$  having analytic singularities does not seem to make (3.1) easier, as in the above way of using Hölder inequality. Now we turn to the exactness of a general Skoda complex and derive it from a special case of (3.1) where  $\beta$  has analytic singularities.

**Proposition 3.4.** *If Conjecture 3.1 is true, then the  $q$ -th Skoda complex (2) is exact for any  $q \geq p$ .*

For this, we will use Demailly's version of Nadel vanishing theorem for a *weakly pseudoconvex* Kähler manifold. A weakly pseudoconvex manifold is a complex manifold which possesses a smooth plurisubharmonic exhaustion function  $\varphi$ . For example, a compact complex manifold is weakly pseudoconvex, taking  $\varphi = 0$ . Also Stein manifolds are weakly pseudoconvex.

**Theorem 3.5** (Nadel vanishing theorem [D, (5.11)]). *Let  $(Y, \omega)$  be a weakly pseudoconvex Kähler manifold and  $F$  a holomorphic line bundle over  $Y$  equipped with a singular hermitian metric  $h$ . Assume that  $\sqrt{-1}\Theta_h(F) \geq \epsilon\omega$  for some continuous positive function  $\epsilon$  on  $Y$ . Then  $H^i(Y, \mathcal{O}(K_Y + F) \otimes J(h)) = 0$  for all  $i \geq 1$ .*

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Boucksom informed me that it can be shown that the special case  $\alpha = \beta$  of (3.1) implies the general cases using the methods of [BFJ].

*Proof of (3.4).* Going back to the Construction of (2.2), first note that  $Y$  is weakly pseudoconvex since it has the pullback under  $f$  of a smooth plurisubharmonic exhaustion function on  $\Omega$  as its own such exhaustion function.  $Y$  is neither compact nor Stein.

We need to show the vanishing of the higher direct images:  $R^i f_*(O_Y(K_{Y/\Omega} \otimes \mathcal{J}(Y, f^*((q-m)\varphi + \psi)))$  for  $i \geq 1$ . Using projection formula and the fact that  $\Omega$  is Stein, it suffices to show the vanishing of  $H^i(Y, O_Y(K_{Y/\Omega} + f^*L) \otimes \mathcal{J}(f^*((q-m)\varphi + \psi)))$  for a sufficiently positive line bundle  $L$  on  $\Omega$ ,  $i \geq 1$  and  $0 \leq j \leq p$ . Taking  $L - K_\Omega$  positive enough, it suffices to show that

$$H^i(Y, O_Y(K_Y + f^*L) \otimes \mathcal{J}(f^*((q-m)\varphi + \psi))) = 0 \quad (3)$$

To show (3), take  $F = f^*(L)$ . We need to construct a singular metric  $h$  of  $f^*(L)$  such that its curvature current is strictly positive and the multiplier ideal sheaf  $\mathcal{J}(Y, h)$  is the same as  $\mathcal{J}(f^*((q-m)\varphi + \psi))$ . This will be shown possible by giving  $h$  as product of a smooth metric of a positive line bundle (which is  $f^*L$  minus  $E$ , sum of small multiples of exceptional divisors to be specified below), the singular metric precisely given by  $E$  for the  $\mathbf{Q}$ -line bundle  $\mathcal{O}(E)$  and  $f^*((q-m)\varphi + \psi)$  (a plurisubharmonic function, which can be seen as a singular metric of  $\mathcal{O}_Y$ ).

We use the following lemma [Vo, Proposition 3.24], which generalizes a result for a point blow-up originally due to Kodaira used in the Kodaira embedding theorem [GH, pp.185-187].

**Lemma 3.6.** *Let  $f_1 : Y_1 \rightarrow Y_0$  be the blow-up of  $Y_0$  along a complex submanifold  $Z_0$  of codimension  $k_0$  and  $E_1$  be the exceptional divisor of  $f_1$  in  $Y_1$ . Let  $A_0$  be any ample line bundle on  $Y_0$ . Then there is a large enough integer  $a > 1$  such that the  $\mathbf{Q}$ -line bundle  $f^*A_0 - \frac{1}{a}E_1$  is positive.*

Now let us suppose that the log-resolution  $f$  is composed of smooth blow-ups  $f_M \circ f_{M-1} \circ \dots \circ f_1$ . Set  $Y_0 := \Omega$  and  $A_0 := L$ . By abuse of notation,  $E_m$  denote all the proper transforms of the exceptional divisor  $E_m$  in  $Y_m$  up to  $Y_M$ . We choose large enough integers  $a_1, \dots, a_M$  as follows.

- $a_1$  is chosen to be large enough to satisfy that  $f_1^*L - \frac{1}{a_1}E_1$  is positive by (3.6).
- $a_2$  is chosen to be large enough so that :

$$f_2^*f_1^*L = f_2^*\left(\underbrace{f_1^*L - \frac{1}{a_1}E_1}_{\text{positive}} + \frac{1}{a_1}E_1\right) = f_2^*\left(\underbrace{\left(f_1^*L - \frac{1}{a_1}E_1\right) - \frac{1}{a_2}E_2 + \frac{1}{a_2}E_2}_{\text{positive}} + f_2^*\left(\frac{1}{a_1}E_1\right)\right)$$

- $a_3$  is chosen to be large enough so that :

$$f_3^*f_2^*f_1^*L = f_3^*\left(\underbrace{\dots}_{\text{positive}} - \frac{1}{a_3}E_3 + \frac{1}{a_3}E_3 + f_3^*\left(\frac{1}{a_2}E_2 + f_2^*\left(\frac{1}{a_1}E_1\right)\right)\right)$$

where the last three terms are rewritten as  $\frac{1}{a_3}E_3 + \frac{1}{a_2}f_3^*E_2 + \frac{1}{a_1}f_3^*f_2^*E_1$ .

- Similarly for  $a_m$  ( $m \geq 4$ ):

$$f_4^*f_3^*f_2^*f_1^*L = (\text{a positive line bundle}) + \frac{1}{a_4}E_4 + \frac{1}{a_3}f_4^*E_3 + \frac{1}{a_2}f_4^*f_3^*E_2 + \frac{1}{a_1}f_4^*f_3^*f_2^*E_1.$$

$\vdots$

$$f^*L = f_M^* \dots f_1^*L = (\text{positive}) + \frac{1}{a_M}E_M + \frac{1}{a_{M-1}}f_M^*E_{M-1} + \dots + \frac{1}{a_1}f_M^*f_{M-1}^* \dots f_2^*E_1.$$

It is now clear that we can take  $E$  to be

$$E := \frac{1}{a_M} E_M + \frac{1}{a_{M-1}} f_M^* E_{M-1} + \cdots + \frac{1}{a_1} f_M^* f_{M-1}^* \cdots f_2^* E_1$$

for large enough integers  $a_1, \dots, a_M$  so that  $\mathcal{J}(\varphi_E + f^*((q-m)\varphi + \psi)) = \mathcal{J}(f^*((q-m)\varphi + \psi))$  ( $\varphi_E$  is the weight function associated to the  $\mathbf{Q}$ -divisor  $E$ ) according to Conjecture 3.1. This proves (3.4).  $\square$

## 4 Work of Hörmander and Skoda on $L^2$ estimates

The Skoda division theorem of [Sk72] is an answer to the following natural *division problem*. Let  $X$  be a smooth projective variety. Let  $f \in H^0(X, K_X + H + qG)$  and  $g_1, \dots, g_p \in H^0(X, G)$  be sections of the appropriate line bundles so that we may want to divide  $f$  by  $g_1, \dots, g_p$ . Of course, the main point is to give the right conditions under which we can do so (we refer to [Sk72], [D], [L] and others for the precise statements of the Skoda division theorem). First, Hörmander [H67] considered a very special case of this problem in a local setting. He applied his  $L^2$  methods for  $\bar{\partial}$  operator [H65] in the setting of a double complex as in (6). The division problem itself is concerned only with the multiplication map  $P : \mathcal{H}_0^1 \rightarrow \mathcal{H}_0^0$  in the notation of (6), but his method requires to consider all the other Hilbert spaces and operators in (6) to the left and to down.

The innovation of [Sk72] made it possible not to use such a double complex at least for the division problem itself. Since we will use the  $L^2$  estimates for  $\bar{\partial}$  as in [Sk72] in the next section, we recall the two fundamental lemmas at the end of this section after we first briefly recall the fundamental applications of Hörmander's  $L^2$  methods for  $\bar{\partial}$  operator in algebraic geometry.

The most important instances of such applications are the vanishing theorems of (Kawamata-Viehweg-)Nadel type, the extension theorems of Ohsawa-Takegoshi type and the division theorems of Skoda type. In each of these, (with some simplification for the exposition) we first set up a  $\bar{\partial}$  equation on a complex manifold  $X$ . Let the  $\bar{\partial}$  equation be

$$\bar{\partial}x = y \tag{4}$$

where  $y$  is given and  $x$  is the unknown. More precisely,  $y$  is an  $L$ -valued  $(n, 1)$  form and  $x$  is an  $L$ -valued  $(n, 0)$  form where  $L$  is a line bundle on  $X$ .

- For Nadel vanishing theorem (see, for example, [G]), the RHS of (4) is a  $\bar{\partial}$ -closed representative of the Dolbeault cohomology class in  $H^{n,1}(X, L)$  which we want to show to be  $\bar{\partial}$ -exact.
- For Ohsawa-Takegoshi extension (see, for example, [D]), we take a  $\bar{\partial}$  equation (4) such that the RHS is a  $\bar{\partial}$ -exact form  $y := \bar{\partial}\sigma$  where  $\sigma$  is an ‘easy’ extension (of a given holomorphic section on a subvariety  $Z$  we want to extend). The problem of  $\sigma$  is that it is not holomorphic and is not coming with the right estimates. The solution of (4) will give us  $\sigma - x$  which is holomorphic and we can arrange it so that  $x|_Z = 0$ .
- For Skoda division (see, for example, [Sk72], [D]), again we take a  $\bar{\partial}$  equation whose RHS is a  $\bar{\partial}$ -exact form  $\bar{\partial}\sigma =: y$  where  $\sigma$  is an ‘easy’ division. More precisely, to divide  $f$  by  $g_1, \dots, g_p$ , take  $\sigma = (h_1, \dots, h_p)$  where  $h_i = \frac{1}{|g|^2} \bar{g}_i f$  for  $1 \leq i \leq p$ . Then clearly  $f = \sum g_i h_i$ . Again, the problem of  $\sigma$  is that it is not holomorphic and is not coming with the right estimates.

Continuing for Skoda division, at this point, in principle one could go ahead and solve the  $\bar{\partial}$  equation (4) in the standard way involving three Hilbert spaces  $\mathcal{H}_0^1, \mathcal{H}_1^1, \mathcal{H}_2^1$  of (6). Instead of this, [H67] solves a slightly different  $\bar{\partial}$  equation which involves the diagram chasing in (6) and necessitates to consider the whole double complex. Skoda [Sk72] came up with yet another method which turned out to be the most natural and efficient one. It involves the four Hilbert spaces of the following functional analysis lemma.

**Lemma 4.1.** *Let  $\mathcal{E}_0, \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$  be Hilbert spaces. Let  $P : \mathcal{F}_0 \rightarrow \mathcal{E}_0$  be a bounded operator. Let  $T : \mathcal{F}_0 \rightarrow \mathcal{F}_1$  and  $S : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  be unbounded, closely defined operators such that  $S \circ T = 0$ . Let  $\mathcal{G} \subset \mathcal{E}_0$  be a closed subspace such that  $P(\text{Ker } T) \subset \mathcal{G}$ . We have  $P(\text{Ker } T) = \mathcal{G}$  if and only if there exists a constant  $C > 0$  such that*

$$\|P^*u + T^*\beta\|^2 + \|S\beta\|^2 \geq C\|u\|^2$$

for all  $u \in \mathcal{E}_0$  and all  $\beta \in \text{Dom } T^* \cap \text{Dom } S \subset \mathcal{F}_1$ .

Another important ingredient of the  $L^2$  estimates for  $\bar{\partial}$  is the following Bochner-Kodaira inequality, also known as the *basic estimate*. Let  $\Omega \subset \mathbf{C}^n$  be a Stein bounded open subset and  $L$  a line bundle on  $\Omega$ . Let  $e^{-\psi}$  be a singular hermitian metric of  $L$ . Let  $\mathcal{F}_i$  be the Hilbert space of  $L$ -valued  $(0, i)$  forms that are square integrable with respect to  $e^{-\psi}$ . Let  $T : \mathcal{F}_0 \rightarrow \mathcal{F}_1$  and  $S : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  be the  $\bar{\partial}$  operators.

**Lemma 4.2** (Bochner-Kodaira [H65]). *For all  $\beta \in \text{Dom } T^* \cap \text{Dom } S$ , we have*

$$\|T^*\beta\|^2 + \|S\beta\|^2 \geq \int_{\Omega} (\sqrt{-1}\partial\bar{\partial}\psi)(\beta, \beta)e^{-\psi}$$

Here we define  $(\sqrt{-1}\partial\bar{\partial}\psi)(\beta, \beta)$  to be (for  $\beta = \beta^1 d\bar{z}_1 + \cdots + \beta^n d\bar{z}_n$ )

$$(\sqrt{-1}\partial\bar{\partial}\psi)(\beta, \beta) := \sum_{1 \leq p, q \leq n} \frac{\partial^2 \psi}{\partial z_p \partial \bar{z}_q} \beta^p \bar{\beta}^q$$

and regard  $\int_{\Omega} (\sqrt{-1}\partial\bar{\partial}\psi)(\beta, \beta)e^{-\psi}$  as the norm of  $\beta$  with respect to  $\sqrt{-1}\partial\bar{\partial}\psi$ .

*Remark 4.3.* In the use of Hörmander's  $L^2$  estimates for  $\bar{\partial}$  with these lemmas, we need the standard procedure of regularizing the plurisubharmonic weight  $\psi$  by a sequence of smooth plurisubharmonic functions  $(\psi_{\nu})_{\nu \geq 1}$ . For simplicity in notations, here and in the next section, we adopt the convention that each  $\psi$  means the  $\nu$ -th regularized  $\psi_{\nu}$  so that we can take  $\sqrt{-1}\partial\bar{\partial}\psi_{\nu}$  and so on. The resulting holomorphic function  $u_{\nu}$  at each step comes with a uniform bound that is independent of  $\nu$ , so we can take the limit  $u$  as  $\nu \rightarrow \infty$  in the usual way. See [K1] for more on this technical point.

## 5 Proof of Main Theorem

### 5.1 Algebraic preliminaries

Let  $A$  be a commutative ring and  $M$  be the dual of the free module  $M' := A^{\oplus p}$  of rank  $p$ . We view an element of  $\bigwedge^k M$  as an alternating function on  $(v_1, \dots, v_k)$  where  $v_i \in A^{\oplus p}$ . Let  $\varepsilon_1, \dots, \varepsilon_p$  be the basis of  $M'$  and let  $e_1, \dots, e_p$  be the dual basis of  $M$ . Let  $h \in M'$ . Let  $i(h)$  be the contraction by  $h$ , that is (for each  $m \geq 1$ ), the map  $i(h) : \bigwedge^m M \rightarrow \bigwedge^{m-1} M$  determined by



$$(i(h)(\eta))(v_1, \dots, v_{m-1}) = \eta(h, v_1, \dots, v_{m-1})$$

for every  $m$ -form  $\eta \in \bigwedge^m M$ . Then it is well-known that

**Proposition 5.1.** *For every  $l, n \geq 1$  and  $\varphi \in \bigwedge^l M, \psi \in \bigwedge^n M$ , we have*

$$i(h)(\varphi \wedge \psi) = (i(h)\varphi) \wedge \psi + (-1)^l \varphi \wedge (i(h)\psi).$$

*Proof.* It is sufficient to prove the identity for the special case when

$$\begin{aligned} \varphi &= e_{i_1} \wedge \dots \wedge e_{i_l} \quad (1 \leq i_1 < \dots < i_l \leq p) \quad \text{and} \\ \psi &= e_{j_1} \wedge \dots \wedge e_{j_n} \quad (1 \leq j_1 < \dots < j_n \leq p). \end{aligned}$$

It helps to rewrite the indices  $j_1 =: i_{l+1}, \dots, j_n =: i_{l+n}$  and observe that

$$\begin{aligned} i(h)\varphi &= \sum_{k=1}^l (-1)^{k-1} h_{i_k} e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_l} \\ \varphi \wedge (i(h)\psi) &= e_{i_1} \wedge \dots \wedge e_{i_l} \wedge \sum_{k=l+1}^{l+n} (-1)^{k-1} h_{i_k} e_{i_{l+1}} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_{l+n}} \\ i(h)(\varphi \wedge \psi) &= \sum_{k=1}^{l+n} (-1)^{k-1} h_{i_k} e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_{l+n}}. \end{aligned}$$

□

Now taking  $h = g_1 \varepsilon_1 + \dots + g_p \varepsilon_p$ , it is easy to see that the map  $i(h) : \bigwedge^m M \rightarrow \bigwedge^{m-1} M$  is our Koszul map  $P$  of (1). Also we let  $P^\vee$  denote the map  $e(\psi) : \bigwedge^{m-1} M \rightarrow \bigwedge^m M$  given by taking wedge with  $\frac{1}{|g|^2} \psi$  where  $\psi = \overline{g_1} e_1 + \dots + \overline{g_p} e_p$ . That is, for each  $u \in \bigwedge^{m-1} M$ , we have

$$\begin{aligned} P^\vee(u) &= \frac{1}{|g|^2} u \wedge \psi = \frac{1}{|g|^2} \left( \sum_I u_I e_I \right) \wedge (\overline{g_1} e_1 + \dots + \overline{g_p} e_p) \\ &= \frac{1}{|g|^2} \sum_J \sum_{k=1}^m (-1)^{m-k} \overline{g_{j_k}} u_{j_1 \dots \widehat{j_k} \dots j_m} \end{aligned} \tag{5}$$

where the index  $J$  in the first summation denotes and ranges over  $J = (j_1, \dots, j_m)$  where  $1 \leq j_1 < \dots < j_m \leq p$ . The last equality comes from the following argument:  $e_I \wedge e_\ell$  is not zero for  $\ell$  such that  $\{i_1, \dots, i_{m-1}, \ell\}$  has  $m$  elements. Rewriting the set  $\{i_1, \dots, i_{m-1}, \ell\}$  in the increasing order as  $\{j_1, \dots, j_m\}$  where  $1 \leq j_1 < \dots < j_m \leq p$ , we note that

$$e_{i_1} \wedge \dots \wedge e_{i_{m-1}} \wedge e_\ell = (-1)^{m-k} e_{j_1} \wedge \dots \wedge e_{j_m}$$

when  $\ell = j_k$  for some  $1 \leq k \leq m$ .

As a consequence of (5.1), we have (taking  $l = m - 1$ )

**Corollary 5.2.**

1.  $P(P^\vee u) = P^\vee(Pu) + (-1)^{m-1}u$  for all  $u \in \bigwedge^{m-1} M$ .

2.  $P(P^\vee u) = (-1)^{m-1}u$  if  $Pu = 0$ .

since the map  $i(h)\psi : \bigwedge^m M \rightarrow \bigwedge^m M$  is the multiplication by 1.

Now we turn to define our Hilbert spaces and their double complex. For  $i \geq 0$  (though we actually need  $i = 0, 1, 2$  only) and  $0 \leq m \leq p$ , let  $\widetilde{\mathcal{H}}_i^m$  be the Hilbert space completion of the smooth  $((q-m)L + M)$ -valued  $(0, i)$  forms that are square-integrable with respect to  $e^{-(q-m)\varphi} e^{-\psi}$ . Let  $\mathcal{H}_i^m$  be the direct sum of  $\binom{p}{m}$  copies of  $\widetilde{\mathcal{H}}_i^m$  for which we use the basis  $\{e_{i_1} \wedge \cdots \wedge e_{i_m} | 1 \leq i_1 < \cdots < i_m \leq p\}$ . Each  $\mathcal{H}_i^m$  is given the inner product of the direct sum Hilbert space. Let  $T : \mathcal{H}_0^m \rightarrow \mathcal{H}_1^m$  and  $S : \mathcal{H}_1^m \rightarrow \mathcal{H}_2^m$  be the direct sum of  $\bar{\partial}$  operators.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{H}_0^2 & \xrightarrow{P} & \mathcal{H}_0^1 & \xrightarrow{P} & \mathcal{H}_0^0 \longrightarrow 0 \\
T \downarrow & & T \downarrow & & T \downarrow & & T \downarrow \\
\cdots & \longrightarrow & \mathcal{H}_1^2 & \xrightarrow{P} & \mathcal{H}_1^1 & \xrightarrow{P} & \mathcal{H}_1^0 \longrightarrow 0 \\
S \downarrow & & S \downarrow & & S \downarrow & & S \downarrow \\
\cdots & \longrightarrow & \mathcal{H}_2^2 & \xrightarrow{P} & \mathcal{H}_2^1 & \xrightarrow{P} & \mathcal{H}_2^0 \longrightarrow 0 \\
\bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \longrightarrow 0
\end{array} \tag{6}$$

Let each  $P$  be the Koszul map of (1). First we compute  $P^*u$  using the fact that  $(P^*u, v) = (u, Pv)$  for all  $v$ .

**Throughout the rest of this paper**, the index  $I$  denotes  $(i_1, \dots, i_{m-1})$  where  $1 \leq i_1 < \cdots < i_{m-1} \leq p$  and the index  $J$  denotes  $(j_1, \dots, j_m)$  where  $1 \leq j_1 < \cdots < j_m \leq p$ .

**Proposition 5.3.** *If  $Pu = 0$ , then  $\|P^*u\|^2 = \|u\|^2$ .*

*Proof.* Let  $u = \sum_I u_I e_I$  and  $v = \sum_J v_J e_J$ . Then

$$P(v) = \sum_J v_J P(e_J) = \sum_J v_J \sum_{k=1}^m (-1)^{k-1} g_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_m}.$$

From the condition that  $(P^*u, v) = (u, Pv)$  for all  $v \in \mathcal{H}_0^m$ , we can determine  $P^*u$ . Namely, the coefficient for  $v_J$  in the summation  $(u, Pv)$  must be the coefficient for  $e_J$  in  $P^*u$ . Also we take into account the fact that  $(P^*u, v)$  is in  $\mathcal{H}_0^m$  and  $(u, Pv)$  is in  $\mathcal{H}_0^{m-1}$  with one less power of  $\frac{1}{|g|}$  in the weight for the inner product. Thus we have

$$P^*u = \frac{1}{|g|^2} \sum_J x_{j_1 \dots j_m} e_{j_1} \wedge \cdots \wedge e_{j_m} \quad \text{where} \quad x_{j_1 \dots j_m} = \sum_{k=1}^m u_{j_1 \dots \widehat{j_k} \dots j_m} (-1)^{k-1} \overline{g_{j_k}}. \tag{7}$$

From (5), we have  $P^*u = (-1)^{m-1} P^\vee u$ . Then  $\|P^*u\|^2 = (P^*u, P^*u) = (u, P(P^*u)) = (u, (-1)^{m-1} P(P^\vee u)) = (u, u)$  by (5.2).

□

## 5.2 Proof of (1.1)

In the  $q$ -th Skoda complex (2), let  $S_m$  denote the sheaf  $\Lambda^m V \otimes \mathcal{J}((q-m)\varphi + \psi) \otimes \mathcal{O}((q-m)L + M)$  ( $0 \leq m \leq p$ ). We want to show the exactness in the middle of  $S_{m+1} \xrightarrow{P} S_m \xrightarrow{P} S_{m-1}$  for every  $m \geq 0$  (defining  $S_{-1} := 0$ ). Since the exactness of a complex is a local property, it is sufficient to show

$$\text{Im } P|_{\Omega} = \text{Ker } P|_{\Omega} \quad \text{in } H^0(\Omega, S_m)$$

where  $\Omega \subset X$  be a Stein open subset.

To apply the functional analysis lemma (4.1) for this, consider the corresponding Hilbert spaces on  $\Omega$ ,  $P_{m-1} : \mathcal{H}_0^{m-1} \rightarrow \mathcal{H}_0^{m-2}$  and  $P_m : \mathcal{H}_0^m \rightarrow \mathcal{H}_0^{m-1}$ . In the setting of (4.1), take  $P := P_m$ ,  $\mathcal{E}_0 := \mathcal{H}_0^{m-1}$ ,  $\mathcal{F}_i := \mathcal{H}_i^m$  ( $i = 0, 1, 2$ ) and take  $\mathcal{G} := \text{Ker } P_{m-1}$  in  $\mathcal{H}_0^{m-1}$ . It suffices to show that  $P_m(\text{Ker } T) = \mathcal{G}$ . Now consider

$$\|P^*u + T^*\beta\|^2 + \|S\beta\|^2 = \|T^*\beta\|^2 + \|S\beta\|^2 + \|P^*u\|^2 + 2\text{Re}(P^*u, T^*\beta). \quad (8)$$

First, note that  $\|P^*u\|^2 = \|u\|^2$  from (5.3). Our plan toward having  $C\|u\|^2$  as in (4.1) is to divide  $2\text{Re}(P^*u, T^*\beta)$  into a  $u$  part and a  $\beta$  part. Then the  $\beta$  part being less than  $\|T^*\beta\|^2 + \|S\beta\|^2$  will finish the proof. More precisely, now apply the Bochner-Kodaira inequality (4.2) to get

$$\|T^*\beta\|^2 + \|S\beta\|^2 \geq \int_{\Omega} \sum_J \left( (q-m)\sqrt{-1}\partial\bar{\partial}\log|g|^2 + \sqrt{-1}\partial\bar{\partial}\psi \right) (\beta_J, \beta_J) e^{-\varphi_1} dV$$

where  $dV$  is the Lebesgue volume form and  $\varphi_1 := (q-m)\log|g|^2 + \psi$  is the weight of the Hilbert spaces  $\mathcal{H}_i^m$ . Here we have  $(q-m)$  times the norm of  $\beta$  with respect to  $\sqrt{-1}\partial\bar{\partial}\log|g|^2$ . This will be cancelled out by another multiple of the same norm of  $\beta$  coming out of  $2\text{Re}(P^*u, T^*\beta)$ . More precisely, we will show that  $2\text{Re}(P^*u, T^*\beta) \geq -\frac{1}{B}\|u\|^2 - T_1$  where  $B$  is a constant  $B > 1$  which we fix throughout and (see (10))

$$T_1 := B \int_{\Omega} |g|^{-2} \sum_{i_1 < \dots < i_{m-1}} |e^{\varphi} T_{i_1 \dots i_{m-1}}|^2 e^{-\varphi_1} d\lambda$$

is the second term of the RHS of (10). Now our main inequality to show is

$$T_1 \leq (m(p - (m-1)) + 1) \int_{\Omega} \sum_J (\sqrt{-1}\partial\bar{\partial}\log|g|^2) (\beta_J, \beta_J) e^{-\varphi_1} dV.$$

Then we can take

$$q = \max_{0 \leq m \leq p} m(p - m + 1) + 1$$

(which gives  $q$  in (1.1)) so that (Skod $_q$ ) is exact, where we apply (4.1) with  $C = 1 - \frac{1}{B}$ .

Now we begin the main computations following the above outline. In order to consider  $2\text{Re}(P^*u, T^*\beta)$ , we write arbitrary  $u \in \mathcal{H}_0^{m-1}$ ,  $\beta \in \mathcal{H}_1^m$  as

$$\begin{aligned} u &= \sum_{i_1 < \dots < i_{m-1}} u_{i_1 \dots i_{m-1}} e_{i_1} \wedge \dots \wedge e_{i_{m-1}} \\ \beta &= \sum_{j_1 < \dots < j_m} (\beta_{j_1 \dots j_m}^1 d\bar{z}_1 + \dots + \beta_{j_1 \dots j_m}^n d\bar{z}_n) e_{j_1} \wedge \dots \wedge e_{j_m}, \end{aligned}$$

we use (7) to obtain (letting  $\varphi := \log |g|^2$ )

$$\begin{aligned}
2 \operatorname{Re}(P^* u, T^* \beta) &= 2 \operatorname{Re}(\bar{\partial}(P^* u), \beta) = 2 \operatorname{Re} \int_U \sum_{j_1 < \dots < j_m} \mathcal{S}_J e^{-\varphi_1} dV \quad \text{where} \\
\mathcal{S}_J &:= u_{j_2 \dots j_m} \left( \overline{\frac{\partial}{\partial z_1} (g_{j_1} e^{-\varphi}) \beta_{j_1 \dots j_m}^1 + \frac{\partial}{\partial z_2} (g_{j_1} e^{-\varphi}) \beta_{j_1 \dots j_m}^2 + \dots + \frac{\partial}{\partial z_n} (g_{j_1} e^{-\varphi}) \beta_{j_1 \dots j_m}^n} \right) \\
&+ u_{j_1 j_3 \dots j_m} \left( \overline{\frac{\partial}{\partial z_1} (g_{j_2} e^{-\varphi}) \beta_{j_1 \dots j_m}^1 + \frac{\partial}{\partial z_2} (g_{j_2} e^{-\varphi}) \beta_{j_1 \dots j_m}^2 + \dots + \frac{\partial}{\partial z_n} (g_{j_2} e^{-\varphi}) \beta_{j_1 \dots j_m}^n} \right) \\
&+ \dots \dots \dots \\
&+ u_{j_1 \dots j_{m-1}} \left( \overline{\frac{\partial}{\partial z_1} (g_{j_m} e^{-\varphi}) \beta_{j_1 \dots j_m}^1 + \frac{\partial}{\partial z_2} (g_{j_m} e^{-\varphi}) \beta_{j_1 \dots j_m}^2 + \dots + \frac{\partial}{\partial z_n} (g_{j_m} e^{-\varphi}) \beta_{j_1 \dots j_m}^n} \right).
\end{aligned}$$

Then we can rewrite this sum over  $J$  as the sum over  $I$  :

$$2 \operatorname{Re}(\bar{\partial}(P^* u), \beta) = \int_{\Omega} \sum_{i_1 < \dots < i_{m-1}} u_{i_1 \dots i_{m-1}} \overline{T_{i_1 \dots i_{m-1}}} e^{-\varphi_1} dV$$

where (understanding that the index  $I \cup t$  of  $\beta_{I \cup t}$  denotes the rearrangement in the right order as far as  $|I \cup t| = m$ ) we define

$$\overline{T_{i_1 \dots i_{m-1}}} := \sum_{t \notin I, 1 \leq t \leq p} \left( \overline{\frac{\partial}{\partial z_1} (g_t e^{-\varphi}) \beta_{I \cup t}^1 + \frac{\partial}{\partial z_2} (g_t e^{-\varphi}) \beta_{I \cup t}^2 + \dots + \frac{\partial}{\partial z_n} (g_t e^{-\varphi}) \beta_{I \cup t}^n} \right) \quad (9)$$

Remembering that  $|u|^2 := \sum_{i_1 < \dots < i_{m-1}} |u_{i_1 \dots i_{m-1}}|^2$ , we have

$$\begin{aligned}
2 \operatorname{Re}(\bar{\partial}(P^* u), \beta) &\geq -\frac{1}{B} \int_{\Omega} |g|^2 |u|^2 e^{-2\varphi - \varphi_1} dV \\
&\quad - B \int_{\Omega} |g|^{-2} \sum_{i_1 < \dots < i_{m-1}} |e^{\varphi} T_{i_1 \dots i_{m-1}}|^2 e^{-\varphi_1} dV \quad (10)
\end{aligned}$$

where we used the fact that for any complex numbers  $X, Y$ :  $|X|^2 + |Y|^2 + 2 \operatorname{Re}(XY) \geq 0$  and also  $\frac{1}{B} |X|^2 + B |Y|^2 + 2 \operatorname{Re}(XY) \geq 0$  for any  $B \geq 1$ . Then we consider

$$\begin{aligned}
&|e^{\varphi} T_{i_1 \dots i_{m-1}}|^2 \\
&= \left| e^{\varphi} \sum_{t \notin I, 1 \leq t \leq p} \left( \overline{\frac{\partial}{\partial z_1} (g_t e^{-\varphi}) \beta_{I \cup t}^1 + \frac{\partial}{\partial z_2} (g_t e^{-\varphi}) \beta_{I \cup t}^2 + \dots + \frac{\partial}{\partial z_n} (g_t e^{-\varphi}) \beta_{I \cup t}^n} \right) \right|^2 \\
&= \left| \sum_{t \notin I, 1 \leq t \leq p} \left( e^{\varphi} \frac{\partial}{\partial z_1} (g_t e^{-\varphi}) \beta_{I \cup t}^1 + e^{\varphi} \frac{\partial}{\partial z_2} (g_t e^{-\varphi}) \beta_{I \cup t}^2 + \dots + e^{\varphi} \frac{\partial}{\partial z_n} (g_t e^{-\varphi}) \beta_{I \cup t}^n \right) \right|^2 \\
&= |g|^{-4} \left| \sum_{t \notin I, 1 \leq t \leq p} \sum_{k=1}^n \sum_{s=1}^p \overline{g_s} \left( g_s \frac{\partial g_t}{\partial z_k} - g_t \frac{\partial g_s}{\partial z_k} \right) \beta_{I \cup t}^k \right|^2
\end{aligned}$$

noting that (for each  $t$ )

$$e^\varphi \frac{\partial}{\partial z_k} (g_t e^{-\varphi}) = |g|^{-2} \sum_{s=1}^p \overline{g_s} \left( g_s \frac{\partial g_t}{\partial z_k} - g_t \frac{\partial g_s}{\partial z_k} \right). \quad (11)$$

Finally we have the following inequalities:

$$\begin{aligned} & B |g|^{-2} \sum_{i_1 < \dots < i_{m-1}} |e^\varphi T_{i_1 \dots i_{m-1}}|^2 \\ &= B |g|^{-2} \sum_{i_1 < \dots < i_{m-1}} |g|^{-4} \left| \sum_{t \notin I, 1 \leq t \leq p} \sum_{k=1}^n \sum_{s=1}^p \overline{g_s} \left( g_s \frac{\partial g_t}{\partial z_k} - g_t \frac{\partial g_s}{\partial z_k} \right) \beta_{I \cup t}^k \right|^2 \\ &= B |g|^{-6} \sum_{i_1 < \dots < i_{m-1}} \left| \sum_{s=1}^p \overline{g_s} \sum_{\substack{t \notin I \\ 1 \leq t \leq p}} \sum_{k=1}^n \left( g_s \frac{\partial g_t}{\partial z_k} - g_t \frac{\partial g_s}{\partial z_k} \right) \beta_{I \cup t}^k \right|^2 \end{aligned} \quad (12)$$

$$\leq B |g|^{-6} \sum_{i_1 < \dots < i_{m-1}} |g|^2 \left| \sum_{s=1}^p \sum_{\substack{t \notin I \\ 1 \leq t \leq p}} [[s, t]]_{I \cup t} \right|^2 \quad (13)$$

$$\leq (p - (m - 1)) B |g|^{-4} \sum_{i_1 < \dots < i_{m-1}} \sum_{\substack{t \notin I \\ 1 \leq t \leq p}} \left| \sum_{s=1}^p [[s, t]]_{I \cup t} \right|^2 \quad (14)$$

$$\leq m(p - (m - 1)) B |g|^{-4} \sum_J \sum_{1 \leq q < r \leq p} |[[q, r]]_J|^2 \quad (15)$$

$$\leq (m(p - (m - 1)) + 1) \sum_J (\sqrt{-1} \partial \bar{\partial} \log |g|^2) (\beta_J, \beta_J) \quad (16)$$

Here we defined (from (13) on)

$$[[r, s]]_J := \sum_{k=1}^n \left( g_r \frac{\partial g_s}{\partial z_k} - g_s \frac{\partial g_r}{\partial z_k} \right) \beta_J^k$$

for any  $1 \leq r, s \leq p$  and the index  $J$  with  $|J| = m$ . The implications  $(12) \rightarrow (13)$  and  $(13) \rightarrow (14)$  are given by Cauchy-Schwarz while  $(14) \rightarrow (15)$  follows from an elementary counting argument. Also we used that

$$\begin{aligned} (\sqrt{-1} \partial \bar{\partial} \log |g|^2) (\beta_J, \beta_J) &= \sum_{1 \leq p, q \leq n} \frac{\partial^2}{\partial z_p \partial \bar{z}_q} (\log |g|^2) \beta_J^p \bar{\beta}_J^q \\ &= |g|^{-4} \sum_{1 \leq q < r \leq p} \left| \sum_{k=1}^n \left( g_q \frac{\partial g_r}{\partial z_k} - g_r \frac{\partial g_q}{\partial z_k} \right) \beta_J^k \right|^2. \end{aligned}$$

This completes the proof of (1.1).

*Remark 5.4.* The value of  $q$  in Theorem 1.1 can be improved if we have a better inequality between (12) and  $\sum_J(\sqrt{-1}\partial\bar{\partial}\log|g|^2)(\beta_J, \beta_J)$  in (16). For example, suppose that  $p = 2$ : (1.1) says that  $q = \lfloor \frac{1}{4}p^2 + \frac{1}{2}p + \frac{5}{4} \rfloor = 3$  works. However, easy computation shows that (12) is less than 2 times  $\sum_J(\sqrt{-1}\partial\bar{\partial}\log|g|^2)(\beta_J, \beta_J)$  in this case. Therefore (the optimal value)  $q = 2$  also works in this case in (1.1).

## 6 Applications to the local syzygy

As we discussed in the introduction, [LL] used the exactness of a Skoda complex to show that not every integrally closed ideal is a multiplier ideal. Recall the setting from the introduction: let  $\Omega \subset X$  be a connected open subset of a complex manifold  $X$ . Let  $e^{-\varphi}$  be a singular weight on  $\Omega$  where  $\varphi$  is a plurisubharmonic function on  $\Omega$ . From  $e^{-\varphi}$ , there are two fundamental ways to define an ideal sheaf of local holomorphic function germs  $u$ : collecting those with  $|u|^2 e^{-\varphi}$  bounded above locally, on one hand and collecting those with the integral  $\int_{\Omega} |u|^2 e^{-\varphi}$  finite, on the other hand. Let us denote the former by  $\mathcal{I}(\varphi)$  and the latter by  $\mathcal{J}(\varphi)$ . If  $\varphi$  has analytic singularities and of the form  $\varphi = \log(|f_1|^2 + \cdots + |f_p|^2)$ , then  $\mathcal{I}(\varphi)$  is the integral closure of the ideal generated by  $f_1, \dots, f_p$ . It is important to distinguish  $\mathcal{I}(\varphi)$  and  $\mathcal{J}(\varphi)$  clearly, so let us call  $\mathcal{I}(\varphi)$  as the **sublevel** ideal sheaf of  $\varphi$  (while  $\mathcal{J}(\varphi)$  is well known as the *multiplier* ideal sheaf of  $\varphi$ ).

**Proposition 6.1.** *A sublevel ideal sheaf  $\mathcal{I}(\varphi)$  is integrally closed.*

*Proof.* Suppose that a local holomorphic function  $f$  satisfies an equation

$$f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + f_k = 0$$

where  $a_i \in \mathcal{I}(\varphi)^i$ . We have the following elementary bound [D97, Ch.II Lemma 4.10] for the roots of a monic polynomial

$$|f| \leq 2 \max_{1 \leq i \leq k} |a_i|^{\frac{1}{i}}$$

locally. Therefore  $|f|^2 e^{-\varphi}$  is locally bounded above. □

Now we apply our main theorem in the local setting as in [LL]. We only need a slight modification of the statements from [LL] due to the fact that our  $q$  in (1.1) is not optimal as in the algebraic case of (1.1). Let  $X$  be a smooth complex algebraic variety of dimension  $n$  and let  $(\mathcal{O}, \mathfrak{m})$  be the local ring of a point  $x \in X$ . Let  $h_1, \dots, h_p \in \mathfrak{m}$  be any collection of non-zero elements generating an ideal  $\mathfrak{a} \subset \mathcal{O}$ . Our main theorem (1.1) implies the following version of Theorem B in [LL] for which we just note that our exact Skoda complex sits inbetween the two Koszul complexes in the statement.

**Theorem 6.2** (see **Theorem B** [LL]). *Let  $\mathcal{J}(\psi)$  be the multiplier ideal sheaf of a plurisubharmonic function  $\psi$  which is defined in a neighborhood of  $x$ . There exists an integer  $p' \geq p$  such that for every  $0 \leq r \leq p$ , the natural map*

$$H_r \left( K_{\bullet}(h_1, \dots, h_p) \otimes \mathfrak{a}^{p'-r} \mathcal{J}(\psi) \right) \rightarrow H_r \left( K_{\bullet}(h_1, \dots, h_p) \otimes \mathcal{J}(\psi) \right)$$

*vanishes.*

The original Theorem B was stronger when  $\varphi$  is algebraic, saying that we could actually take  $p' = p$  in that case. Now using the isomorphism between  $H_r$  and  $\text{Tor}_r$  and taking  $p = n$ , we have

**Corollary 6.3** (see **Corollary C** [LL]). *There exists an integer  $n' \geq n$  such that the natural maps*

$$\text{Tor}_r(\mathfrak{m}^{n'-r}\mathcal{J}(\psi), \mathbf{C}) \rightarrow \text{Tor}_r(\mathcal{J}(\psi), \mathbf{C})$$

*vanish for all  $0 \leq r \leq n$ .*

**Corollary 6.4** (see **Theorem A** [LL]). *Let  $\mathcal{J} = \mathcal{J}(\psi) \subset \mathcal{O}$  be (the germ at  $x$  of) any multiplier ideal. Then there exists an integer  $n' \geq n$  with the property: For  $p \geq 1$ , no minimal  $p$ th syzygy of  $\mathcal{J}$  vanishes modulo  $\mathfrak{m}^{n'+1-p}$ .*

(6.3) implies (6.4). Take  $n'$  from (6.3). Suppose that a minimal  $p$ th syzygy of  $\mathcal{J}$  vanishes modulo  $\mathfrak{m}^{n'+1-p}$ . That is, given a minimal free resolution, a linear combination of the columns of  $u_p$ , namely  $u_p(e)$  for some  $e \in R_p = \mathcal{O}^{b_p}$  satisfies  $u_p(e) \in \mathfrak{m}^a R_{p-1} = \mathfrak{m}^a \cdot \mathcal{O}^{b_{p-1}}$  where  $a = n' + 1 - p \geq 2$ . Then Proposition 1.1 [LL] says that  $e$  represents a class lying in the image of  $\text{Tor}_p(\mathfrak{m}^{a-1}\mathcal{I}, \mathbf{C}) \rightarrow \text{Tor}_p(\mathcal{I}, \mathbf{C})$ . This contradicts to (6.3).  $\square$

Finally, [LL, Example 2.2] says that there exists an integrally closed ideal  $\mathcal{I}$  supported at a point with a first syzygy vanishing to arbitrary order  $a$  at the origin. It cannot be a multiplier ideal due to (6.4). Hence Corollary 1.2 is proved since it is sufficient to examine at the local ring of a point.

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